NORTH-HOLLAND

A Problem on the Exponent of Primitive Digraphs*

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ABSTRACT

A digraph $D(A)$ is called primitive if and only if A, the (0, 1) connection matrix of $D(A)$, is primitive. The exponent of primitivity of $D(A)$ is defined to be $\gamma(D(A)) = \min\{k \in Z_+ : A^k \gg 0\}$, where Z_+ denotes the set of positive integers. In a recent paper, we have proved the conjecture $\gamma(D(A)) \leqslant (m-1)^2 + 1$ due to Robert E. Hartwig and Michael Neumann, where m is the degree of the minimal polynomial of A. In this paper, we characterize the equality case of the upper bound $\gamma(D(A)) \leqslant (m-1)^2 + 1$.

1. INTRODUCTION AND NOTATION

For all terminology and notation used here we follow [1].

We shall use the notation $i \to j$ and $i \not\to j$ to denote, respectively, that there is a directed edge from vertex i to vertex j, and that there is not. The distance $d(i, j)$ from vertex i to vertex j is the minimal length of a path linking vertex i to vertex j. The symbols D and D_{A^k} denote, respectively, the diameter of $D(A)$ and that of $D(A^k)$. We also designate the letters m and s to denote, respectively, the degree of the minimal polynomial and the length of the shortest circuit in $D(A)$.

$A = (a_{ij})$, the (0, 1) connection matrix of $D(A)$, is defined as follows: $a_{ij} = 1$ if and only if $i \to j$, $a_{ij} = 0$ if and only if $i \not\to j$.

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Some well-known facts concerning nonnegative matrices which we shall use are the following.

$$I + A + \cdots + A^D \gg 0,$$

$$A + A^2 + \cdots + A^{D+1} \gg 0,$$

$$(A^k)_{ij} > 0 \Leftrightarrow i \xrightarrow{k} j \quad \text{in } D(A),$$

$$A^k \gg 0 \Leftrightarrow i \xrightarrow{k} j \quad \text{for any } i, j \in V(D(A)).$$

Thus, in particular, if $(A^k)_{ii} > 0$, then the vertex i lies on a closed path of length k in $D(A)$. Also it is known that $D \leqslant m - 1$.

Suppose $a_1, \dots, a_t$ is a set of distinct positive integers with $\gcd(a_1, \dots, a_t) = 1$. Then we define $\Phi(a_1, \dots, a_t)$ to be the least integer m such that every integer $k \geqslant m$ can be expressed in the form $k = c_1 a_1 + \cdots + c_t a_t$, where $c_1, \dots, c_t$ are some nonnegative integers. A well-known result due to Schur shows that $\Phi(a_1, \dots, a_t)$ is well defined when $\gcd(a_1, \dots, a_t) = 1$.

2. SOME LEMMAS AND THE PRELIMINARY THEOREM

DEFINITION 2.1. The digraph B_n is defined as follows: $V(B_n) = \{1, 2, \dots, n\}$ and $E(B_n) = \{(n-1, 1)\} \cup \{(i, i+1)\colon 1 \leqslant i \leqslant n\}$, where the operation $+$ is taken modulo n.

LEMMA 2.1. *$\gamma(D(A)) = (n-1)^2 + 1$ if and only if $D(A)$ is isomorphic to B_n, where n is the number of the vertices in $D(A)$.*

LEMMA 2.2 [2]. *Let A be an $n \times n$ nonnegative matrix with l distinct eigenvalues. Then $D(A)$ contains a circuit of length not greater than l.*

LEMMA 2.3. *If A is an $m \times n$ matrix, $b = (b_1, b_2, \dots, b_m)^T$, and $X = (x_1, x_2, \dots, x_n)^T$, then:*

(1) *The equation $AX = b$ has a solution if and only if* $\operatorname{rank} A = \operatorname{rank}(A, B)$.

(2) *Suppose $b = (0, 0, \dots, 0)^T$. Then*

if $\operatorname{rank} A = n$, *then $AX = b$ has the sole solution $X = (0, 0, \dots, 0)^T$;*
if $\operatorname{rank} A \leqslant n - 1$, *then $AX = b$ has nonzero solutions.*

THEOREM 2.1. *If $\gamma(D(A)) = (m-1)^2 + 1$, then $D(A)$ must satisfy either of the following two cases:*

(1) $s = D$.
(2) $s = D+1$, $m = D+1$, *and* $A^{D+1} = aA + bI$, *where* $a, b > 0$.

The main idea of the proof of Theorem 2.1 is to check all the theorems and lemmas in [1] to see whether or not the equality case $\gamma(D(A)) = (m-1)^2 + 1$ is accessible. This is not difficult, but the procedure is very long and tedious, so we omit all of the proof.

3. MAIN RESULT

DEFINITION 3.1. Suppose $B, C \subset V(D(A))$. We define the following notation:

$B \Rightarrow C$: $b \to c$ for any $b \in B$ and for any $c \in C$.
$B \not\Rightarrow C$: $b \not\to c$ for any $b \in B$ and for any $c \in C$.
$B \to C$: for any $b_1 \in B$ there exists some $c_1 \in C$ such that $b_1 \to c_1$, and also for any $c_2 \in C$ there exists some $b_2 \in B$ such that $b_2 \to c_2$.

DEFINITION 3.2. A digraph E_{D+1} is defined as follows: $V(E_{D+1}) = \bigcup_{i=0}^{D} A_i$, where every A_i is nonempty, $|A_0| = 1$, and $A_i \cap A_j = \varnothing$ for any $i \neq j$; the only edges of E_{D+1} are $A_D \to A_1$, $A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_D \Rightarrow A_0$.

By Definition 3.2, we know the digraph E_{D+1} is not unique, since $|A_i|$, $1 \leqslant i \leqslant D$, is arbitrary and the edges from A_D to A_1 are not uniquely specified.

THEOREM 3.1. *Suppose $s = D$. Then $\gamma(D(A)) = D^2 + 1$ if and only if $D(A)$ is isomorphic to some digraph E_{D+1}.*

Proof. The proof of sufficiency is trivial, and so we just prove the necessity.

Since $\gamma(D(A)) = D^2 + 1$, there exist two vertices $j, k \in V(D(A))$ such that $j \overset{D^2}{\not\to} k$. It is easy to prove that neither j nor k lies on a closed path whose length is D. Suppose $a_1 \to a_2 \to \cdots \to a_D \to a_1$ is a closed path whose length is D, so for any $1 \leqslant i \leqslant D$ there exist x_i and y_i such that

$$1 \leqslant x_i, y_i \leqslant D \quad \text{and} \quad j \xrightarrow{x_i} a_i \xrightarrow{y_i} k.$$

Let $p_i = x_i + y_i$.

If $p_i \leqslant D$ for some $1 \leqslant i \leqslant D$, since $\Phi(D+1, D) + p_i \leqslant D(D+1) + D = D^2$, then $j \xrightarrow{D^2} k$; this is a contradiction. If $D + 2 \leqslant p_i \leqslant 2D$ for some $1 \leqslant i \leqslant D$, since $p_i + (p_i - D - 2)D + (2D - p_i)(D+1) = D^2$, then $j \xrightarrow{D^2} k$, also a contradiction. Therefore we have $p_i = x_i + y_i = D + 1$ for any $1 \leqslant i \leqslant D$ and any x_i, y_i such that $1 \leqslant x_i, y_i \leqslant D$ and $j \xrightarrow{x_i} a_i \xrightarrow{y_i} k$.

Next we will prove there exists some l such that $1 \leqslant l \leqslant D$ and $x_l = 1$. Let $x_l = \min\{x_i\colon 1 \leqslant i \leqslant D\}$. If $x_l \geqslant 2$, then $y_l = D + 1 - x_l \leqslant D - 1$. Note that

$$j \xrightarrow{x_{l-1}} a_{l-1} \rightarrow a_l \xrightarrow{\quad y_l \quad} k, \qquad \text{i.e.,} \quad j \overset{1+x_{l-1}+y_l}{\longrightarrow} k,$$

and $1 < 1 + x_{l-1} + y_l \leqslant 1 + x_{l-1} + D - 1 \leqslant 2D$, so $1 + x_{l-1} + y_l = D + 1 = x_l + y_l$, i.e., $x_{l-1} = x_l - 1$, contradicting the choice of x_l.

Without loss of generality, now we suppose $x_1 = 1$. For any $0 \leqslant i \leqslant D - 1$ we can prove $x_{i+1} = x_i + 1 = i + 1$, $y_{i+1} = D + 1 - (i+1) = D - i$, and

$$j \overset{x}{\nrightarrow} a_{i+1} \qquad \text{for any} \quad x \neq x_{i+1} \text{ such that } 1 \leqslant x \leqslant D + i. \tag{3.1}$$

For any $1 \leqslant i \leqslant D$ we define $A_i = \{b \in V(D(A))\colon d(j, b) = i\}$. By (3.1) we know $d(j, a_i) = i$, i.e., $a_i \in A_i$, where $1 \leqslant i \leqslant D$.

For any $b_i \in A_i$ and any l, $1 \leqslant l \leqslant D$, such that $b_i \xrightarrow{l} a_{i+1}$ (if $i + 1 = D + 1$, we define $a_{D+1} = k$), since

$$j \xrightarrow{i} b_i \xrightarrow{l} a_{i+1}, \qquad \text{i.e.,} \quad j \xrightarrow{i+l} a_{i+1},$$

by (3.1) we have $l = 1$, i.e., $A_i \Rightarrow a_{i+1}$ for any $1 \leqslant i \leqslant D$. For any $b_{i-1} \in A_{i-1}$ (if $i - 1 = 0$, we define $A_0 = \{j\}$) and any l, $1 \leqslant l \leqslant D$, such that $b_{i-1} \xrightarrow{l} b_i$, since

$$j \xrightarrow{i-1} b_{i-1} \xrightarrow{\quad l \quad} b_i \rightarrow a_{i+1}, \qquad \text{i.e.,} \quad j \xrightarrow{i+l} a_{i+1},$$

by (3.1) we have $l = 1$, i.e., $A_{i-1} \Rightarrow A_i$. Therefore we have proved $j = A_0 \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_D \Rightarrow k$.

If $k \in A_1$, since $1 + (D+1)(D-1) = D^2$, then $j \xrightarrow{D^2} k$, which is a contradiction, so $k \notin A_1$. If $k \in A_i$ for some i such that $2 \leqslant i \leqslant D$, then

$$j \xrightarrow{i-1} a_{i-1} \rightarrow k,$$

i.e., $x_{i-1} + y_{i-1} = i - 1 + 1 = i \neq D + 1$, which is also a contradiction. So $k \notin A_i$ for any $1 \leqslant i \leqslant D$; then we have $j = k$. (If $j \neq k$, then $d(j,k) > D$, as $k \notin \bigcup_{i=1}^{D} A_i$—a contradiction.)

For any $b_i, b_i' \in A_i$, $1 \leqslant i \leqslant D$, such that $b_i \neq b_i'$, if $b_i \rightarrow b_i'$, then

$$j \xrightarrow{i} b_i \rightarrow a_{i+1} \quad \text{and} \quad j \xrightarrow{i} b_i \rightarrow b_i' \rightarrow a_{i+1},$$

contradicting (3.1). Therefore we have $A_i \not\Rightarrow A_i$ for any $1 \leqslant i \leqslant D$.

Since the length of the shortest circuit is $s = D$, by (3.1) it is easy to prove $A_l \not\Rightarrow A_m$ for any l and m such that $1 \leqslant l$, $m \leqslant D$ and $l \not\equiv m - 1 \pmod D$.

Note that $a_D \rightarrow a_1$. If $b_D \not\Rightarrow A_1$ for some $b_D \in A_D$, then we can prove $b_D \neq a_D$ and $d(b_D, a_D) > D$, which is a contradiction. Similarly, if $A_D \not\Rightarrow b_1$ for some $b_1 \in A_1$, then $b_1 \neq a_1$ and $d(a_1, b_1) > D$, also a contradiction. Therefore $A_D \rightarrow A_1$, and we complete the proof of Theorem 3.1. ■

THEOREM 3.2. *Suppose $s = D$. Then $\gamma(D(A)) = (m-1)^2 + 1$ if and only if $D(A)$ is isomorphic to B_n.*

Proof. The proof of sufficiency is trivial, and so we just prove the necessity.

In [1, Theorem 3.4], we have proved $\gamma(D(A)) \leqslant D^2 + 1$ if $s = D$. Since $\gamma(D(A)) = (m-1)^2 + 1$, we have $m - 1 = D$ and $\gamma(D(A)) = D^2 + 1$. By Theorem 3.1, $D(A)$ is isomorphic to E_{D+1}.

Now we prove $|A_i| = 1$ for any $1 \leqslant i \leqslant D$. If there exists some i, $1 \leqslant i \leqslant D$, such that $|A_i| \geqslant 2$, then we can suppose $\{j, k\} \subset A_i$. Let $e(j)$ be the usual jth unit column space. By the definition of E_{D+1}, it is easy to prove at least one of the following two cases is true:

(1) $Ae(j) = Ae(k)$;
(2) $e(j)^T A = e(k)^T A$.

Then $\det A = 0$ is always true, so $\lambda = 0$ is an eigenvalue of A, and the minimal polynomial of A is $A^{D+1} = a_D A^D + a_{D-1}A^{D-1} + \cdots + a_1 A$, from which we have $(A^{D+1})_{aa} = a_D(A^D)_{aa}$. This can't be true, because $a \in A_0$ lies on a closed path with the length of $D + 1$, but a doesn't lie on a closed path with the length D. Therefore $|A_i| = 1$ for any $1 \leqslant i \leqslant D$, from which $D(A)$ is isomorphic to B_n. ■

THEOREM 3.3. *Suppose $s = D + 1$, $m = D + 1$, and $A^{D+1} = aA + bI$, where $a, b > 0$. Then $\gamma(D(A)) = (m-1)^2 + 1$ if and only if $D(A)$ is isomorphic to the complete graph K_n $(n \geqslant 3)$ and $A^2 = (n-2)A + (n-1)I$.*

Proof. The proof of sufficiency is trivial, and so we just prove the necessity.

Suppose $D(A)$ is not isomorphic to K_n, i.e., $D \geqslant 2$. Since $s = m = D + 1$, by Lemma 2.2, A has $D + 1$ distinct eigenvalues, i.e., Spec $A = \{\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}, \bar{\lambda}_{r+1}, \ldots, \lambda_{r+t}, \bar{\lambda}_{r+t}\}$, where $r + 2t = D + 1$, λ_i is a real eigenvalue for any $1 \leqslant i \leqslant r$, λ_{r+i} and $\bar{\lambda}_{r+i}$ are conjugate complex eigenvalues for any $1 \leqslant i \leqslant t$, λ_1 is a Perron-Frobenius eigenvalue, and the multiplicities of $\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}, \bar{\lambda}_{r+1}, \ldots, \lambda_{r+t}, \bar{\lambda}_{r+t}$ in the characteristic polynomial are, respectively, $1, n_2, \ldots, n_r, n_{r+1}, n'_{r+1}, \ldots, n_{r+t}, n'_{r+t}$. Since $A^{D+1} = aA + bI$ and $b > 0$, every eigenvalue of A is nonzero. Note that $s = D + 1$, so trace$(A^k) = 0$ for all $k = 1, 2, \ldots, D$, i.e.,

$$
\begin{aligned}
&\lambda_1 + \cdots + n_r\lambda_r + n_{r+1}\lambda_{r+1} + n'_{r+1}\bar{\lambda}_{r+1} + \cdots \\
&\qquad\qquad + n_{r+t}\lambda_{r+t} + n'_{r+t}\bar{\lambda}_{r+t} = 0, \\
&\lambda_1^2 + \cdots + n_r\lambda_r^2 + n_{r+1}\lambda_{r+1}^2 + n'_{r+1}\bar{\lambda}_{r+1}^2 + \cdots \\
&\qquad\qquad + n_{r+t}\lambda_{r+t}^2 + n'_{r+t}\bar{\lambda}_{r+t}^2 = 0, \qquad (3.2) \\
&\qquad\qquad \vdots \\
&\lambda_1^D + \cdots + n_r\lambda_r^D + n_{r+1}\lambda_{r+1}^D + n'_{r+1}\bar{\lambda}_{r+1}^D + \cdots \\
&\qquad\qquad + n_{r+t}\lambda_{r+t}^D + n'_{r+t}\bar{\lambda}_{r+t}^D = 0.
\end{aligned}
$$

Since all λ_i's are distinct, it is obvious, by real polynomial theory, that $n_{s+i} = n'_{s+i}$ for any $1 \leqslant i \leqslant t$, and then Equation (3.2) is

$$\begin{aligned} \lambda_1 + \cdots + n_r \lambda_r + n_{r+1}\left(\lambda_{r+1} + \bar{\lambda}_{r+1}\right) + n_{r+2}\lambda_{r+2} & \\ + n_{r+2}\bar{\lambda}_{r+2} + \cdots + n_{r+t}\lambda_{r+t} + n_{r+t}\bar{\lambda}_{r+t} &= 0, \\ \lambda_1^D + \cdots + n_r \lambda_r^D + n_{r+1}\left(\lambda_{r+1}^D + \bar{\lambda}_{r+1}^D\right) + n_{r+2}\lambda_{r+2}^D & \\ + n_{r+2}\bar{\lambda}_{r+2}^D + \cdots + n_{r+t}\lambda_{r+t}^D + n_{r+t}\bar{\lambda}_{r+t}^D &= 0. \end{aligned} \tag{3.3}$$

By Lemma 2.3, we have

$$\operatorname{rank}\begin{pmatrix} \lambda_1 & \cdots & \lambda_r & \lambda_{r+1} + \bar{\lambda}_{r+1} & \lambda_{r+2} & \bar{\lambda}_{r+2} & \cdots & \lambda_{r+t} & \bar{\lambda}_{r+t} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^D & \cdots & \lambda_r^D & \lambda_{r+1}^D + \bar{\lambda}_{r+1}^D & \lambda_{r+2}^D & \bar{\lambda}_{r+2}^D & \cdots & \lambda_{r+t}^D & \bar{\lambda}_{r+t}^D \end{pmatrix}$$

$$< D,$$

i.e.,

$$\begin{aligned} 0 &= \det\begin{pmatrix} \lambda_1 & \cdots & \lambda_r & \lambda_{r+1} + \bar{\lambda}_{r+1} & \lambda_{r+2} & \bar{\lambda}_{r+2} & \cdots & \lambda_{r+t} & \bar{\lambda}_{r+t} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^D & \cdots & \lambda_r^D & \lambda_{r+1}^D + \bar{\lambda}_{r+1}^D & \lambda_{r+2}^D & \bar{\lambda}_{r+2}^D & \cdots & \lambda_{r+t}^D & \bar{\lambda}_{r+t}^D \end{pmatrix} \\ &= \det\begin{pmatrix} \lambda_1 & \cdots & \lambda_r & \lambda_{r+1} & \lambda_{r+2} & \bar{\lambda}_{r+2} & \cdots & \bar{\lambda}_{r+t} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^D & \cdots & \lambda_r^D & \lambda_{r+1}^D & \lambda_{r+2}^D & \bar{\lambda}_{r+2}^D & \cdots & \bar{\lambda}_{r+t}^D \end{pmatrix} \\ &\quad + \det\begin{pmatrix} \lambda_1 & \cdots & \lambda_r & \bar{\lambda}_{r+1} & \lambda_{r+2} & \bar{\lambda}_{r+2} & \cdots & \bar{\lambda}_{r+t} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^D & \cdots & \lambda_r^D & \bar{\lambda}_{r+1}^D & \lambda_{r+2}^D & \bar{\lambda}_{r+2}^D & \cdots & \bar{\lambda}_{r+t}^D \end{pmatrix} \\ &= R\left(\lambda_1, \ldots, \lambda_r, \lambda_{r+2}, \bar{\lambda}_{r+2}, \ldots, \bar{\lambda}_{r+t}\right)\left(R_1 + R_2\right), \end{aligned}$$

where $R(\lambda_1, \dots, \lambda_r, \lambda_{r+2}, \bar{\lambda}_{r+2}, \dots, \bar{\lambda}_{r+t}) \neq 0$,

$$R_1 = \lambda_{r+1} \prod_{\substack{\lambda_i \neq \lambda_{r+1}, \bar{\lambda}_{r+1} \\ \lambda_i \in \operatorname{Spec} A}} (\lambda_{r+1} - \lambda_i),$$

$$R_2 = \bar{\lambda}_{r+1} \prod_{\substack{\lambda_i \neq \lambda_{r+1}, \bar{\lambda}_{r+1} \\ \lambda_i \in \operatorname{Spec} A}} (\bar{\lambda}_{r+1} - \lambda_i),$$

and $R_1 + R_2 = 0$. Note the conjugate complex eigenvalues in Spec A appear as pairs; thus

$$\bar{R}_1 = \bar{\lambda}_{r+1} \prod_{\substack{\lambda_i \neq \lambda_{r+1}, \bar{\lambda}_{r+1} \\ \lambda_i \in \operatorname{Spec} A}} \left(\bar{\lambda}_{r+1} - \bar{\lambda}_i\right) = \bar{\lambda}_{r+1} \prod_{\substack{\lambda_i \neq \lambda_{r+1}, \bar{\lambda}_{r+1} \\ \lambda_i \in \operatorname{Spec} A}} \left(\bar{\lambda}_{r+1} - \lambda_i\right) = R_2,$$

so we have $R_1 = r_1\sqrt{-1}$, where r_1 is real and $r_1 \neq 0$, since $|R_1| \neq 0$.

On the other hand, since $\lambda_i \in$ Spec A is the root of the equation $x^{D+1} - ax - b = 0$, by Vieta's theorem,

$$\begin{aligned} \lambda_1 + \cdots + \lambda_r + \left(\lambda_{r+1} + \bar{\lambda}_{r+1}\right) + \lambda_{r+2} + \bar{\lambda}_{r+2} + \cdots \\ + \lambda_{r+t} + \bar{\lambda}_{r+t} = 0, \\ \vdots \\ \lambda_1^{D-1} + \cdots + \lambda_r^{D-1} + \left(\lambda_{r+1}^{D-1} + \bar{\lambda}_{r+1}^{D-1}\right) + \lambda_{r+2}^{D-1} + \bar{\lambda}_{r+2}^{D-1} + \cdots \\ + \lambda_{r+t}^{D-1} + \bar{\lambda}_{r+t}^{D-1} = 0. \end{aligned} \tag{3.4}$$

The equation (3.3) minus the equation (3.4) then gives

$$\begin{aligned} (n_2 - 1)\lambda_2 + \cdots + (n_r - 1)\lambda_r + (n_{r+1} - 1)\left(\lambda_{r+1} + \bar{\lambda}_{r+1}\right) \\ + (n_{r+2} - 1)\lambda_{r+2} + \cdots + (n_{r+t} - 1)\bar{\lambda}_{r+t} = 0, \\ \vdots \\ (n_2 - 1)\lambda_2^{D-1} + \cdots + (n_r - 1)\lambda_r^{D-1} + (n_{r+1} - 1)\left(\lambda_{r+1}^{D-1} + \bar{\lambda}_{r+1}^{D-1}\right) \\ + (n_{r+2} - 1)\lambda_{r+2}^{D-1} + \cdots + (n_{r+t} - 1)\bar{\lambda}_{r+t}^{D-1} = 0. \end{aligned} \tag{3.5}$$

If $n_i = 1$ for all $2 \leqslant i \leqslant r + t$, then $m = n$, where n is the number of vertices in $V(D(A))$, so $s = D + 1 = m = n$, which can't be true because of the primitivity of $D(A)$. So $n_i - 1$ cannot all be 0; by Lemma 2.3 as above, we have

$$0 = \det\begin{pmatrix} \lambda_2 & \cdots & \lambda_r & \lambda_{r+1} + \overline{\lambda}_{r+1} & \lambda_{r+2} & \overline{\lambda}_{r+2} & \cdots & \lambda_{r+t} & \overline{\lambda}_{r+t} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_2^{D-1} & \cdots & \lambda_r^{D-1} & \lambda_{r+1}^{D-1} + \overline{\lambda}_{r+1}^{D-1} & \lambda_{r+2}^{D-1} & \overline{\lambda}_{r+2}^{D-1} & \cdots & \lambda_{r+t}^{D-1} & \overline{\lambda}_{r+t}^{D-1} \end{pmatrix}$$

$$= R'\left(\lambda_2, \ldots, \lambda_r, \lambda_{r+2}, \overline{\lambda}_{r+2}, \ldots, \overline{\lambda}_{r+t}\right)\left(R'_1 + R'_2\right),$$

where

$$R'_1 = \lambda_{r+1} \prod_{\substack{\lambda_i \neq \lambda_1, \lambda_{r+1}, \overline{\lambda}_{r+1} \\ \lambda \in \operatorname{Spec} A}} \left(\lambda_{r+1} - \lambda_i\right),$$

$$R'_2 = \overline{\lambda}_{r+1} \prod_{\substack{\lambda_i \neq \lambda_1, \lambda_{r+1}, \overline{\lambda}_{r+1} \\ \lambda_i \in \operatorname{Spec} A}} \left(\overline{\lambda}_{r+1} - \lambda_i\right),$$

$$R'\left(\lambda_2, \ldots, \lambda_r, \lambda_{r+2}, \overline{\lambda}_{r+2}, \ldots, \overline{\lambda}_{r+t}\right) \neq 0, \quad \text{and} \quad R'_1 + R'_2 = 0.$$

Similarly we have $\overline{R'}_1 = R'_2$, so $R'_1 = r_2\sqrt{-1}$, where r_2 is a nonzero real. Then we have $r_1/r_2 = R_1/R'_1 = \lambda_{r+1} - \lambda_1$, but this is a contradiction, since the left side is real while the right is not. Thus $D = 1$, and Theorem 3.3 follows. ∎

REMARK. In the proof of necessity, there is no need to use the condition $\gamma(D(A)) = (m-1)^2 + 1$.

Combining Theorem 2.1, Theorem 3.2, and Theorem 3.3, we have

MAIN THEOREM. *Suppose m is the degree of the minimal polynomial of A, the $(0,1)$ connection matrix of $D(A)$. Then $\gamma(D(A)) = (m-1)^2 + 1$ if and only if $D(A)$ is isomorphic to either of the following two graphs*:

(1) B_n $(n \geqslant 2)$,
(2) K_n $(n \geqslant 3)$.

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